

Economics of Information

LECTURE 6

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Competitive markets and equilibrium

competitive equilibrium

a situation in which prices are such that all arbitrage profits are eliminated

- is it possible for a competitive economy to always be in equilibrium?
- no, because arbitrageurs make no return from their costly activity

“the assumptions that all markets, including that for information, are always in equilibrium and always perfectly arbitrated are inconsistent when arbitrage is costly”

[Grossman and Stiglitz (1980), p. 393]

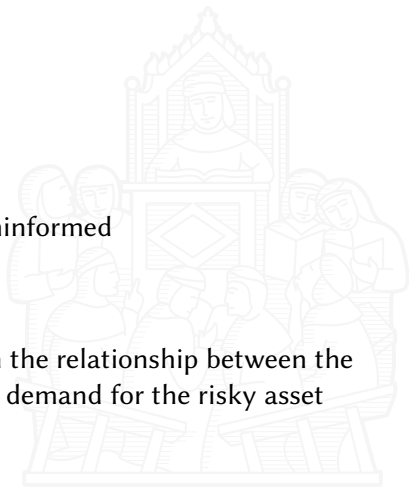


Grossman and Stiglitz (1980)

- market with an '*equilibrium amount of disequilibrium*'
- price reflects information only *partially*
- those who acquire costly information do receive compensation
- those who don't, partially receive information anyway
- when the informed anticipate the return is going to be high, they bid its price up
- conversely when they anticipate the return is going to be low
- informativeness of the price system depends on # informed
- # informed individuals is endogenous
- if prices fully reflect information, then there is no equilibrium

Preliminary assumptions

- two assets are traded in a market
 - a *safe* asset, yielding constant return R
 - a *risky* asset, yielding return $u = \theta + \varepsilon$
- both θ and ε are random variables
- θ is observable at a cost $c > 0$ while ε is unobservable
- *ex ante* identical individuals choose to be informed or uninformed
- demand of informed traders depends on θ and price P
- demand of uninformed traders only depends on P
- uninformed traders have rational expectations and learn the relationship between the distribution of return and the price, and use this in their demand for the risky asset
- random variable x is the supply of risky asset
- uninformed traders cannot observe x



equilibrium price

- define λ as the percentage of informed traders
- equilibrium is $P_\lambda(\theta, x)$ such that demand equals supply
- uninformed are prevented from learning θ by observing $P_\lambda(\theta, x)$
- they cannot distinguish between variations in P due to θ or x
- depending on relative variances of underlying random variables, price reveals some of informed traders' information to uninformed traders

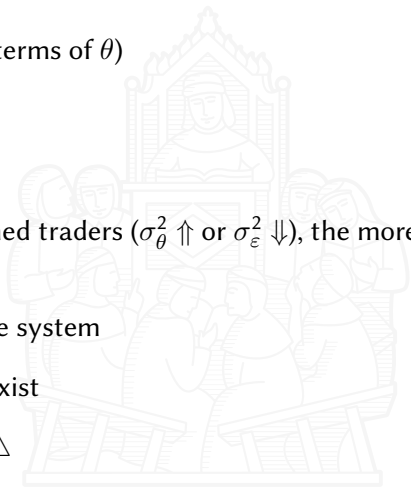
overall equilibrium

- requires λ such that $EU_I = EU_U$



Conjectures

- 1 the higher λ , the more informative the price system (in terms of θ)
- 2 the higher λ , the lower EU_I/EU_U
- 3 the higher c , the lower λ
- 4 the higher the quality of information available to informed traders ($\sigma_\theta^2 \uparrow$ or $\sigma_\varepsilon^2 \downarrow$), the more informative the price system
- 5 the higher the noise ($\sigma_x^2 \uparrow$), the less informative the price system
- 6 in the limit $\sigma_x^2 \rightarrow 0$ a competitive equilibrium does *not* exist
- 7 markets are thinner when λ is close to the extremes of Δ



The model (cont'd)

- trader i is endowed with $\bar{M}_i$ of safe asset and $\bar{X}_i$ of risky asset
- safe asset price equals 1; risky asset price is P
- budget constraint

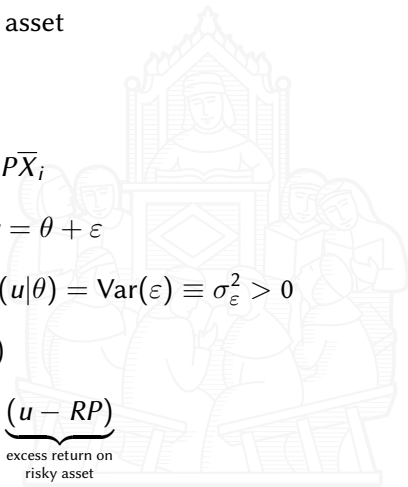
$$PX_i + M_i = W_{0i} \equiv \bar{M}_i + P\bar{X}_i$$

- riskless asset pays \$ return R ; risky asset pays \$ return $u = \theta + \varepsilon$

$$(\theta, \varepsilon) \sim \mathcal{N}(\cdot), \quad E[\varepsilon] = 0, \quad E[\theta\varepsilon] = 0, \quad \text{Var}(u|\theta) = \text{Var}(\varepsilon) \equiv \sigma_\varepsilon^2 > 0$$

- end of period wealth for trader holding portfolio (M_i, X_i)

$$W_{1i} = RM_i + uX_i = RW_{0i} + X_i \underbrace{(u - RP)}_{\text{excess return on risky asset}}$$



The model (cont'd)

- utility function of trader i is CARA with coefficient of absolute risk aversion a

$$V(W_{1i}) = -e^{-aW_{1i}} \quad a > 0$$

- traders maximise expected utility using available information and consider buying signal θ
- they compare max EU of being informed with max EU of being uninformed
- since W_{1i} is a linear function of $\varepsilon \sim \mathcal{N}(\cdot)$ then W_{1i} is normal conditional on θ
- from the definition of lognormal distribution, if $\rho \sim \mathcal{N}(\mu, \sigma^2)$ then $E[e^{-a\rho}] = e^{-a\mu + \frac{a^2}{2}\sigma^2}$
- expected utility of *informed* individual with information θ

$$\begin{aligned} E[V(W_{1i})|\theta] &= -\exp\left\{-aE[W_{1i}|\theta] + \frac{a^2}{2}\text{Var}[W_{1i}|\theta]\right\} = \dots \\ &= -\exp\left\{-a[RW_{0i} + X_i(\theta - RP)] + \frac{a^2}{2}X_i^2\sigma_\varepsilon^2\right\} \end{aligned}$$



The model (cont'd)

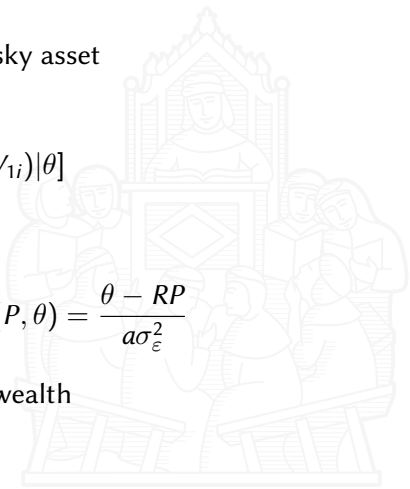
- define X_I as the informed individual's demand for the risky asset
- informed individual's maximisation problem

$$X_I(P, \theta) = \arg \max_{X_I} E[V(W_{1i})|\theta]$$

- FOC

$$\frac{\partial E[V(W_{1i})|\theta]}{\partial X_I} = 0 \quad \Rightarrow \quad X_I(P, \theta) = \frac{\theta - RP}{a\sigma_\varepsilon^2}$$

- as usual with CARA utility, demand doesn't depend on wealth
- there is no subscript i on the left hand side



The model (cont'd)

- let $P^*(\theta, x)$ be a generic price function such that $(u, P^*) \sim \mathcal{N}(\cdot)$
- expected utility of *uninformed* individual

$$\begin{aligned} E[V(W_{1i})|P^*] &= -\exp \left\{ -a E[W_{1i}|P^*] + \frac{a^2}{2} \text{Var}[W_{1i}|P^*] \right\} \\ &= -\exp \left\{ -a [RW_{0i} + X_i(E[u|P^*] - RP)] + \frac{a^2}{2} X_i^2 \text{Var}[u|P^*] \right\} \end{aligned}$$

- define X_U as the uninformed individual's demand for the risky asset

$$\begin{aligned} X_U(P, P^*) &= \arg \max_{X_U} E[V(W_{1i})|P^*] = \frac{E[u|P^*] - RP}{a \text{Var}[u|P^*]} \\ &= \frac{E[u|P^*(\theta, x) = P] - RP}{a \text{Var}[u|P^*(\theta, x) = P]} \end{aligned} \quad (\text{rational expectations})$$



The model (cont'd)

equilibrium price distribution

- define λ as the fraction of traders who decide to become informed
- a price equilibrium is a function $P_\lambda(\theta, x)$ such that $\forall (\theta, x)$ per capita demands of the risky asset equals supply

$$\lambda \cdot X_I(P_\lambda(\theta, x), \theta) + (1 - \lambda) \cdot X_U(P_\lambda(\theta, x), P_\lambda^*) = x$$

- $P_\lambda(\theta, x)$ is a *statistical* equilibrium
- over time uninformed traders observe realisations of (u, P_λ^*) and learn joint distribution
- after learning ceases, demand and expectations are consistent, such that the joint distribution of (u, P_λ^*) persists over time
- market clearing price takes into account the fact that uninformed traders have learned that it contains information



Theorem 1.

if (θ, ε, x) has non-degenerate joint normal distribution such that θ, ε , and x are mutually independent, then there exist equilibrium solution

$$P_{\lambda}(\theta, x) = \alpha_1 + \alpha_2 w_{\lambda}(\theta, x)$$

where $\alpha_1 \in \mathbb{R}$, $\alpha_2 > 0$, and

$$w_{\lambda}(\theta, x) = \begin{cases} \theta - \frac{a\sigma_{\varepsilon}^2}{\lambda}(x - E[x]) & \text{if } \lambda > 0 \\ x & \text{if } \lambda = 0 \end{cases}$$

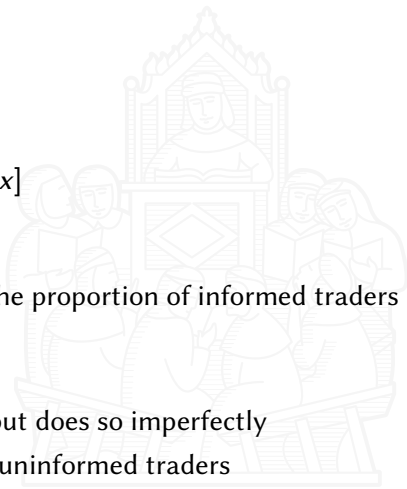
The model (cont'd)

- signal w_λ is a mean-preserving spread of θ , i.e.

$$E[w_\lambda|\theta] = \theta$$

$$\text{Var}[w_\lambda|\theta] = \frac{a^2 \sigma_\varepsilon^4}{\lambda^2} \text{Var}[x]$$

- P_λ is informationally equivalent to w_λ
- the magnitude of the noise is inversely proportional to the proportion of informed traders λ , and proportional to the variance of ε
- if $\lambda = 0$ the price contains no information about θ
- if $\lambda > 0$ the price system conveys information about θ , but does so imperfectly
- the supply noise x prevents w_λ from fully revealing θ to uninformed traders



The model (cont'd)

- **Q:** how well-informed can uninformed traders become from observing P_λ ?
- measure is given by $\text{Var}[w_\lambda|\theta]$
- if $\text{Var}[w_\lambda|\theta] = 0$ then w_λ and θ are perfectly correlated
- $\frac{\partial \text{Var}[w_\lambda|\theta]}{\partial \text{Var}[x]} = \frac{a^2 \sigma_\varepsilon^4}{\lambda^2} > 0 \implies$ the larger the noise, the worse the precision

- $\frac{\partial \text{Var}[w_\lambda|\theta]}{\partial a} > 0; \quad \frac{\partial \text{Var}[w_\lambda|\theta]}{\partial \sigma_\varepsilon^2} > 0; \quad \frac{\partial \text{Var}[w_\lambda|\theta]}{\partial \lambda} < 0$

when individuals are low risk averse ($a \downarrow$), or information is very precise ($\sigma_\varepsilon^2 \downarrow$), or more traders are informed ($\lambda \uparrow$), informed traders' demand is more responsive to θ

- consequently, the price is more informative about θ to uninformed traders
- information is transferred from informed traders to uninformed traders



The model (cont'd)

- what we have characterized so far is the equilibrium price distribution for given λ

equilibrium in the information market

define an *overall* equilibrium as the pair (λ, P_λ^*) such that

$$\begin{cases} EU_I = EU_U & \text{if } 0 < \lambda < 1 & \text{at } P_\lambda^* \\ EU_I < EU_U & \text{if } \lambda = 0 & \text{at } P_0^* \\ EU_I > EU_U & \text{if } \lambda = 1 & \text{at } P_1^* \end{cases}$$



The model (cont'd)

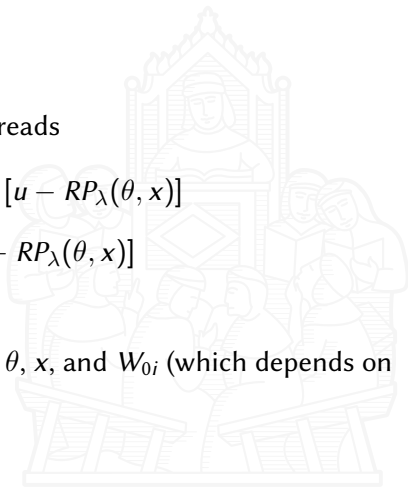
- recall that $W_{1i} = RW_{0i} + X_i(u - RP)$
- end of period *maximised* wealth at the *equilibrium price* reads

$$W_{1I}^\lambda \equiv R(W_{0i} - c) + X_I(P_\lambda(\theta, x), \theta) [u - RP_\lambda(\theta, x)]$$

$$W_{1U}^\lambda \equiv RW_{0i} + X_U(P_\lambda(\theta, x), P_\lambda^*) [u - RP_\lambda(\theta, x)]$$

where c is the cost of observing a realisation of θ

- end of period wealth is random due to randomness of u , θ , x , and W_{0i} (which depends on $P_\lambda(\theta, x)$ and on $\bar{X}_i$ that can be assumed random as well)



Theorem 2.

under the assumptions of **Theorem 1.**, if $\bar{X}_i$ is independent of (u, θ, x) , then

$$\frac{E[V(W_{1I}^\lambda)]}{E[V(W_{1U}^\lambda)]} = e^{ac} \sqrt{\frac{\text{Var}[u|\theta]}{\text{Var}[u|w_\lambda]}} \equiv \gamma(\lambda)$$

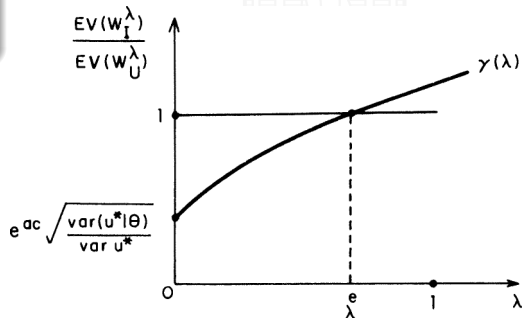
- e.g. if $0 < \lambda < 1$ then overall (internal) equilibrium requires $\gamma(\lambda) = 1$
- $\text{Var}[u|\theta]$ is inversely related to the precision of the information available to informed traders
- $\text{Var}[u|w_\lambda]$ is inversely related to the informativeness of the price system

The model (cont'd)

Corollary

$\gamma(\lambda)$ is a strictly monotone increasing function of λ

- e.g. if $\lambda \uparrow \rightarrow \text{Var}[u|w_\lambda] \downarrow$
(more information is revealed) $\rightarrow \gamma \uparrow$
- this looks counterintuitive but we defined utility as negative !
- when $\lambda = 0$ variance of the signal is maximised $\text{Var}[u|w_\lambda] = \text{Var}[u]$



The model (cont'd)

Theorem 3.

- if $0 < \lambda < 1$, $\gamma(\lambda) = 1$, and $P_\lambda^* = \alpha_1 + \alpha_2 w_\lambda$, then (λ, P_λ^*) is an overall equilibrium
- if $\gamma(\lambda = 1) < 1$, then $(1, P_1^*)$ is an overall equilibrium
- if $\gamma(\lambda = 0) > 1$, then $(0, P_0^*)$ is an overall equilibrium
- for all price equilibria P_λ^* which are monotone functions of w_λ , there exists a unique overall equilibrium (λ, P_λ^*)

Proof

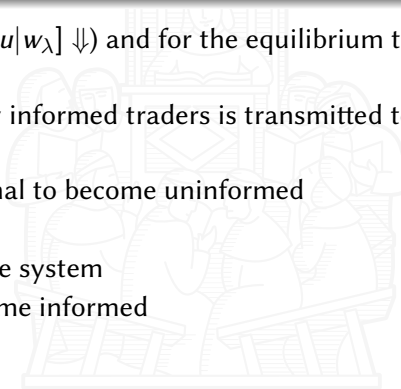
- existence follows from the definition of overall equilibrium
- uniqueness follows from monotonicity of $\gamma(\lambda)$ (cf. **Corollary**)



Price cannot fully reflect costly information

suppose c is small enough so that it is worthwhile for a trader to become informed when no other trader is informed. If there is no noise ($\sigma_x^2 = 0$) then there doesn't exist an equilibrium

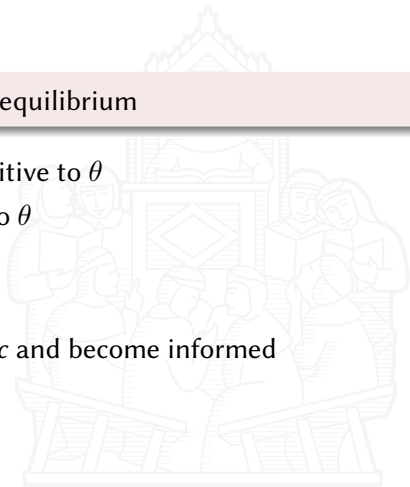
- if $\sigma_x^2 \downarrow$ then informativeness of price system grows ($\text{Var}[u|w_\lambda] \downarrow$) and for the equilibrium to be maintained it must be that $\lambda \downarrow$
- in the limit for $\sigma_x^2 \rightarrow 0$, all the information purchased by informed traders is transmitted to uninformed traders by the price system
- each informed trader, acting as a price taker, finds optimal to become uninformed
- $\lambda > 0$ is not an equilibrium
- but if nobody is informed, no trader learns from the price system
- if c is low, uninformed traders have an incentive to become informed
- $\lambda = 0$ is not an equilibrium
- $\exists$ discontinuity in $\frac{EU_I}{EU_U}$ between $\lambda = 0$ and $\lambda > 0$



Price cannot fully reflect costly information (cont'd)

if information is perfect ($\sigma_\varepsilon^2 = 0$), then there doesn't exist an equilibrium

- as $\sigma_\varepsilon^2 \rightarrow 0$ informed traders' demands become very sensitive to θ
- then also market clearing price becomes very sensitive to θ
- all traders individually desire to be uninformed
- $\lambda > 0$ is not an equilibrium
- but when everyone is uninformed, it pays off to expend c and become informed
- $\lambda = 0$ is not an equilibrium



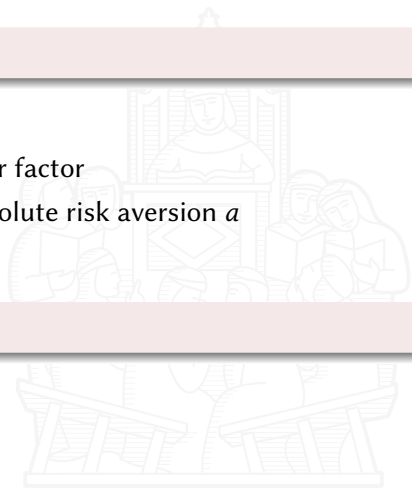
On the thinness of speculative markets

■ why does trade take place?

- differences in beliefs, endowments, preferences
- Grossman (1975, 1977, 1978): preferences are not a major factor
- in our model all traders have the same coefficient of absolute risk aversion a
- assume all traders have the same endowments $(\bar{M}_i, \bar{X}_i)$

■ how much trade takes place?

- beliefs tend to be homogeneous whenever
 - $\lambda \rightarrow 0$ (all traders are uninformed)
 - $\lambda \rightarrow 1$ (all traders are informed)



On the thinness of speculative markets (cont'd)

Theorem

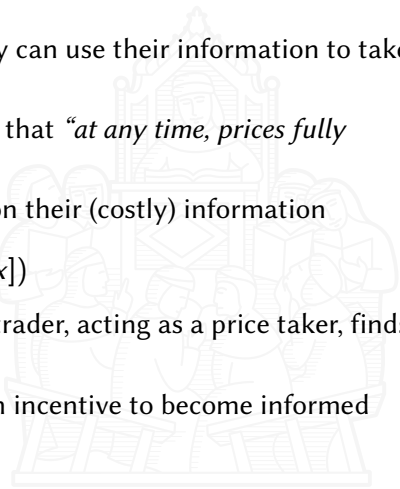
- the amount of trade is a normally distributed random variable which depends on θ and x
- for sufficiently large or small c , the mean and variance of trade is zero
- as the precision of informed traders' information goes to infinity, the mean and variance of trade go to zero

“the result that competitive equilibrium is incompatible with informationally efficient markets should be interpreted as meaning that speculative markets where prices reveal a lot of information will be very thin because it will be composed of individuals with very similar beliefs” [ibidem, p. 403]



On the possibility of perfect markets

- traders have an incentive to become informed only if they can use their information to take ‘better’ positions with respect to uninformed traders
- the *efficient market hypothesis* (EMH, Fama, 1970) asserts that “*at any time, prices fully reflect all available information*”
- if this was true, informed traders couldn’t earn a return on their (costly) information
- the price system reveals the signal $w_\lambda \equiv \theta - \frac{a\sigma_\varepsilon^2}{\lambda}(x - E[x])$
- when EMH is true (e.g. $\sigma_\varepsilon^2 = 0$ or $\sigma_x^2 = 0$) each informed trader, acting as a price taker, finds it optimal to become uninformed
- but when all traders are uninformed, each of them has an incentive to become informed
- no equilibrium exists



On the possibility of perfect markets (cont'd)

Fama (1970)

$c = 0$ is a *sufficient* condition for prices to reveal all available information

Grossman and Stiglitz (1980)

$c = 0$ is a *necessary* condition for prices to reveal all available information

- but this is a *reductiō ad absurdum* !
- competitive markets are important *only* when information is costly ! (cf. Hayek)



On the possibility of perfect markets (cont'd)

- there *cannot* be complete markets
- if there were, competitive equilibrium would not exist
- it is only because of costly transactions and incomplete markets that competitive equilibrium can be established
- there is a fundamental conflict between the efficiency with which markets spread information and incentives to acquire information

“as soon as the assumptions of the conventional perfect capital markets model are modified to allow even a slight amount of information imperfection and a slight cost of information, the traditional theory becomes untenable”

[ibidem, p. 404]

